

The line is part of a circle

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1 Introduction

A mathematical object of interest X is sometimes a subobject $X \rightarrowtail Y$ where Y has structure which makes reasoning about X simpler. For example, the real open interval $(0, 1)$ can be embedded inside the circle $[0, 1]/(0 \sim 1)$, which is a compact space although $(0, 1)$ is not. Note also that the real numbers \mathbb{R} can be embedded into the complex numbers $\mathbb{R} \rightarrowtail \mathbb{C}$ where \mathbb{C} is an algebraically closed field and \mathbb{R} is not. Today, we look at a categorical example of this phenomena, the *Yoneda embedding*.

2 Equivalences of categories and embeddings

Recall that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is **faithful** if for all pairs of objects (X, Y) in \mathcal{C} the function

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\longrightarrow \text{Hom}_{\mathcal{D}}(FX, FY) \\ f &\longmapsto Ff \end{aligned}$$

is injective. Also, if this function is surjective then F is **full**.

Moreover, if for every object $D \in \mathcal{D}$ there exists an object $C \in \mathcal{C}$ such that $FC \cong D$ then F is **essentially surjective**.

This was introduced in Lecture 3 and we mentioned that it provides sufficient conditions for F to be part of the data of an *equivalence of categories*. At the time, we did not have the language of natural transformations, and so we did not give the definition, now though we can.

Definition 2.0.1. An **equivalence of categories** is a pair (F, G) of functors

$$\begin{aligned} F: \mathcal{C} &\longrightarrow \mathcal{D} \\ G: \mathcal{D} &\longrightarrow \mathcal{C} \end{aligned}$$

along with a pair of natural isomorphisms $\eta: \text{id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon: FG \Rightarrow \text{id}_{\mathcal{D}}$.

Exercise 2.0.2. Prove that (F, G) is an equivalence of categories if and only if F is a fully, faithful, essentially surjective functor.

Exercise 2.0.3 (Hard exercise). Does your proof for the previous exercise hold in the first order theory of ZFC sets?

Thus, we can think of a fully, faithful functor (not necessarily essentially surjective) as an embedding of one category into another. The main result of today's lecture will exhibit such an embedding.

3 An introduction to universal properties

In life, it is more important how an object *behaves* than it is what the object *is*. For instance, when pegging in a tent peg, I might use a rock as a hammer. Since the rock in that moment *behaved* like a hammer, does it really matter that what I had was a rock and strictly speaking *not* a hammer?

Mathematically, we can take the same approach.

Definition 3.0.1. Let X, Y be two sets. A **product** of X, Y consists of a set $X \times Y$ along with two functions $\pi_X: X \times Y \rightarrow X, \pi_Y: X \times Y \rightarrow Y$ which together satisfy the following property: if $f: U \rightarrow X, g: U \rightarrow Y$ are any two functions, then there exists a unique function $h: U \rightarrow X \times Y$ which makes the following diagram commute.

$$\begin{array}{ccccc} X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\ & \swarrow \pi_X & \uparrow h \\ & & U & \searrow \pi_Y & \end{array} \quad (1)$$

An example of a product is the cartesian product

$$(x, y) \in X \times Y \Leftrightarrow x \in X \text{ and } y \in Y \quad (2)$$

Proof. Let $f : U \rightarrow X, g : U \rightarrow Y$ be arbitrary. First we prove uniqueness. Say an appropriate $h : U \rightarrow X \times Y$ exists. Then for any $u \in U$, the first entry of $h(u)$ is given by $\pi_X h(u) = f(u)$, and the second entry is given by $\pi_Y h(u) = g(u)$. This means $h(u) = (f(u), g(u))$ which we note is independent of h . We notice that this proves existence too. \square

We notice that Definition 3.0.1 never defined what the set $X \times Y$ of a product *is*, but only defined a *property* of it.

This definition generalises to arbitrary categories immediately.

Definition 3.0.2. A **product** (if it exists) of two objects X, Y in a category \mathcal{C} consists of an object $X \times Y$ along with a pair of morphisms $\pi_X : X \times Y \rightarrow X, \pi_Y : X \times Y \rightarrow Y$ which together satisfy the following property: if $f : U \rightarrow X, g : U \rightarrow Y$ are any two functions, then there exists a unique morphism $h : U \rightarrow X \times Y$ which makes the following diagram commute.

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 & \swarrow f & \uparrow h & \searrow g & \\
 & & U & &
 \end{array} \tag{3}$$

Lemma 3.0.3. *If a product $(X \times Y, \pi_X, \pi_Y)$ exists, then it is unique up to unique isomorphism.*

Proof. Let $(X \hat{\times} Y, \rho_X, \rho_Y)$ be another product. Construct the following diagram, considering only the solid arrows for now.

$$\begin{array}{ccccc}
 & & X \times Y & & \\
 & \swarrow \pi_X & \uparrow j & \searrow \pi_Y & \\
 X & & X \times Y & & Y \\
 & \swarrow \rho_X & \downarrow h & \searrow \rho_Y & \\
 & & X \hat{\times} Y & &
 \end{array} \tag{4}$$

The pairs of morphisms π_X, π_Y and ρ_X, ρ_Y each satisfy the other product's universal property. Thus we obtain two induced morphisms $h : X \times Y \rightarrow X \hat{\times} Y, j : X \hat{\times} Y \rightarrow X \times Y$ which makes the above diagram commute, considering all arrows now.

Figure 1: Yoneda Lemma core idea

$$\begin{array}{ccc}
 \text{Hom}(A, A) & \xrightarrow{\text{Hom}(A, f)} & \text{Hom}(A, X) \\
 \downarrow \Phi_A & & \downarrow \Phi_X \\
 F(A) & \xrightarrow{Ff} & F(X)
 \end{array}$$

$$\begin{array}{ccc}
 \text{id}_A & \xrightarrow{\quad} & f \\
 \downarrow & & \downarrow \\
 u & \xrightarrow{\quad} & (Ff)u = \Phi_X(f)
 \end{array}$$

The composition hj makes the following diagram commute

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 & \searrow f & \uparrow h & \nearrow hj & \\
 & & X \times Y & &
 \end{array} \tag{5}$$

and so does the identity morphism $\text{id}_{X \times Y}$. By uniqueness of such morphisms, we have $hj = \text{id}_{X \times Y}$. A similar argument shows that $jh = \text{id}_{X \times Y}$. \square

4 The Yoneda Lemma

The following Lemma has been referred to as the only theorem in category theory.

Lemma 4.0.1. *Let \mathcal{C} be a small category (that is, a category whose collection of objects is a set), and let $P : \mathcal{C} \rightarrow \underline{\text{Set}}$ be a functor. For any object $A \in \mathcal{C}$ there is a natural bijection*

$$\begin{aligned}
 \Phi : \text{Nat}(\text{Hom}(A, _), P) &\cong P(A) \\
 \eta &\longmapsto \eta_A(\text{id}_A)
 \end{aligned}$$

Proof. The core idea is the diagram shown in Figure 1. We notice that this proves injectivity and surjectivity. \square

Exercise 4.0.2. Finish the proof of Lemma 4.0.1 by proving naturality. If you need a hint see [1].

Definition 4.0.3. A **contravariant functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ is an assignment of an object $FC \in \mathcal{D}$ to every object $C \in \mathcal{C}$ along with a function for each pair of objects (X, Y) in \mathcal{C}

$$Ff : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(FY, FX) \quad (6)$$

(Note the change of order of X, Y), subject to the following conditions:

- For any pair of morphisms $f : X \rightarrow Y, g : Y \rightarrow Z$ in \mathcal{C} we have $F(g \circ f) = F(f) \circ F(g)$,
- For any object $X \in \mathcal{C}$ we have $F(\text{id}_X) = \text{id}_{FX}$.

Exercise 4.0.4. Show that the data of a contravariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is equivalent to the data of a functor $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$.

Exercise 4.0.5. Show that there is a “contravariant” version of Yoneda’s Lemma too. That is, prove the following.

Lemma 4.0.6. *Let \mathcal{C} be a small category and $P : \mathcal{C} \rightarrow \mathcal{D}$ a contravariant functor. For any object $A \in \mathcal{C}$ there is a natural bijection*

$$\text{Nat}(\text{Hom}(_, A), P) \cong P(A)$$

In the special case where $P = \text{Hom}(_, B)$ for some object $B \in \mathcal{C}$, Yoneda’s lemma implies the following natural isomorphism.

$$\text{Nat}(\text{Hom}(_, A), \text{Hom}(_, B)) \cong \text{Hom}(A, B) \quad (7)$$

That is, there is an embedding of categories:

$$\mathcal{C} \rightarrow \underline{\text{Set}}^{\mathcal{C}^{\text{op}}} \quad (8)$$

Facts which we will not prove:

- $\underline{\text{Set}}^{\mathcal{C}^{\text{op}}}$ admits all products.
- $\underline{\text{Set}}^{\mathcal{C}^{\text{op}}}$ admits all coproducts.
- $\underline{\text{Set}}^{\mathcal{C}^{\text{op}}}$ admits all limits and colimits.

- $\underline{\text{Set}}^{\mathcal{C}^{\text{op}}}$ admits all exponential objects and a subobject classifier. In fact, $\underline{\text{Set}}^{\mathcal{C}^{\text{op}}}$ is a *topos*.

All of these holds whether \mathcal{C} has any of this structure or *none*. In fact, even more can be said, we know what the excess in $\underline{\text{Set}}^{\mathcal{C}^{\text{op}}}$ is:

Proposition 4.0.7. *Every object $P \in \underline{\text{Set}}^{\mathcal{C}^{\text{op}}}$ is a colimit of elements in the image of \mathcal{C} under the yoneda embedding.*

Suggestion: somebody makes a talk out of this.

References

- [1] W. Troiani, *Course notes for the Séminaire étudiant de théorie des catégories*, <https://williamtroiani.github.io/CategoryTheory/Lecture6.pdf>