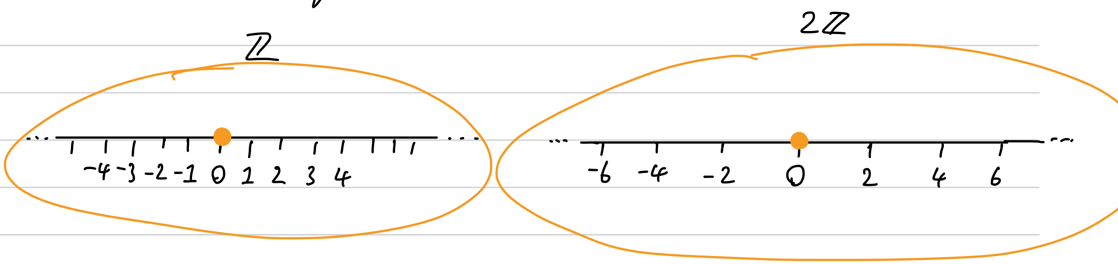
SOC Lecture 2 Notes

Question

- What is the right notion of same-ness of mathematical objects?

  Structure Preservation

Consider "the integers"



$2\mathbb{Z}$ looks just like a stretched, relabelled version of $\mathbb{Z}$

- Object identities within a structure are less important than the relationships between them
- Stretching expresses a way to transition (morph) one structure into another without disturbing its essential nature (ordering).

Def^n: A <u>group</u> $(G, \circ, e)$ consists of a set G, an operation

$$\circ : G \times G \longrightarrow G$$

and an identity element $e \in G$, such that

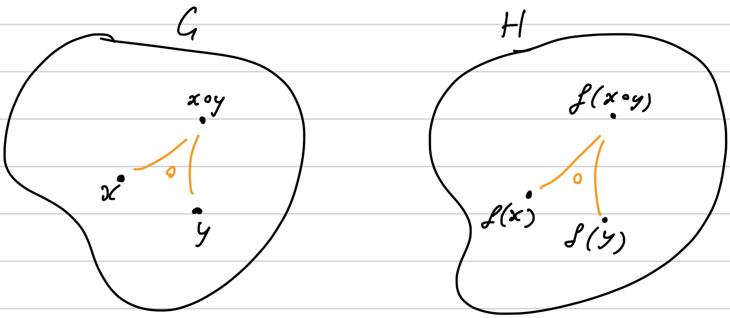
$\forall x, y, z \in G \quad (x \circ y) \circ z = x \circ (y \circ z)$ (Associativity)

$\forall x \in G \quad e \circ x = x \circ e = x$ (Identity)

$\forall x \in G \ \exists y \in G \quad x \circ y = y \circ x = e$ (Inverses)

<u>Def^n</u> A function between groups $f : G \to H$ is a <u>homomorphism</u> iff

$$\forall x, y \in G \quad f(x \circ_G y) = f(x) \circ_H f(y)$$



$$f(x) \circ f(y) = f(x \circ y)$$

<u>Example</u>

$$f : (\mathbb{Z}, +, 0) \longrightarrow (2\mathbb{Z}, +, 0)$$

$$a \longmapsto 2a$$

$$\left(\begin{aligned} f(a+b) &= 2 \cdot (a+b) \\ &= 2a + 2b \\ &= f(a) + f(b) \end{aligned} \right)$$

is a group homomorphism. (so is $a \mapsto -2a$)

$$g : (\mathbb{R}, +, 0) \longrightarrow (\mathbb{R}^+, \cdot, 1)$$

$$x \longmapsto e^x$$

is a group homomorphism $\left(e^{x+y} = e^x \cdot e^y \right)$

Def^{n} A morphism $f: A \to B$ in a category is an **isomorphism** if there exists $g: B \to A$ such that

$$f \circ g = id_B \qquad g \circ f = id_A$$

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[loop right, "id_B"] \end{tikzcd}$$

Notice that preservation of structure is essential

Example $\quad f: \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$

$$(x, y) \longmapsto 2x + y$$

$$\begin{aligned} (0,0) &\longrightarrow 0 \\ (0,1) &\longrightarrow 1 \\ (1,0) &\longrightarrow 2 \\ (1,1) &\longrightarrow 3 \end{aligned}$$

f is bijective as a function

But facts about $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ don't translate into $\mathbb{Z}/4\mathbb{Z}$-facts

$$\begin{array}{ccc} (0,1) & \longleftrightarrow & 1 \\ + & & + \\ (0,1) & \longleftrightarrow & 1 \\ \| & & \| \\ (0,0) & \longleftarrow \times \longrightarrow & 2 \end{array}$$

$f((0,1) + (0,1)) = 2 =$

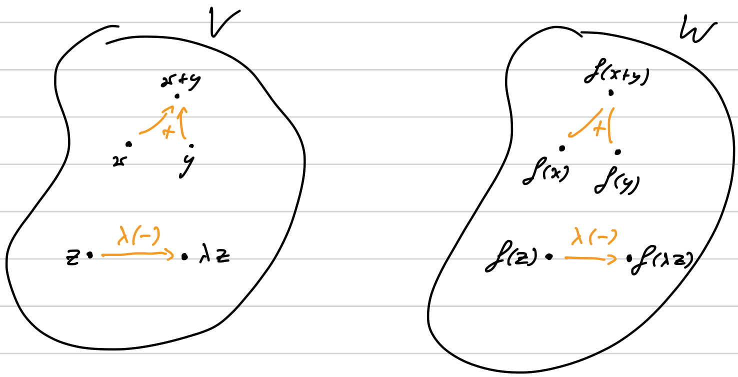
"There exists an element, when doubled, gives the identity"

We can usually guess definitions

- Groups have: multiplication, identity element, so morphisms of groups should preserve them

$$f(x \circ y) = f(x) \circ f(y) \qquad (f : G \to H)$$
$$f(e_G) = e_H$$

- Vector spaces have addition, scalar multiplication



$$f(x+y) = f(x) + f(y)$$
$$f(\lambda z) = \lambda f(z)$$

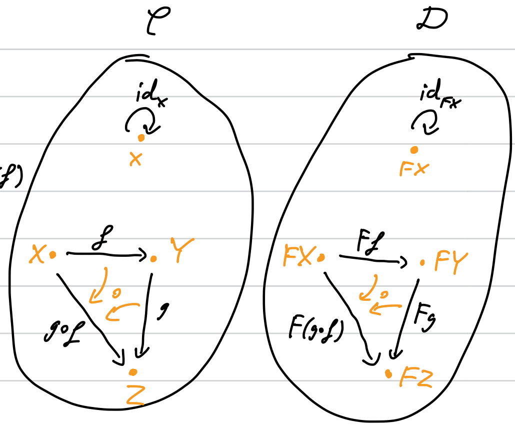
- categories have: composition, identity morphisms

Def^n: A <u>functor</u> $F : \mathcal{C} \to \mathcal{D}$ between categories consist of

- A function $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$
- For every pair of objects $X, Y \in Ob(\mathcal{C})$, a function

$$F_{X,Y} : Hom(X,Y) \to Hom(FX, FY)$$ satisfying...

- $F(id_X) = id_{FX}$
- If $f: X \to Y$ and $g: Y \to Z$ in $\mathcal{C}$ then $F(g \circ f) = F(g) \circ F(f)$



Examples

(1) $F : Set \longrightarrow Set$

$$X \longmapsto X \amalg \{*\}$$

$$(f : X \to Y) \longmapsto \left(\begin{array}{l} Ff : X \amalg \{*\} \to Y \amalg \{*\} \\ \quad x \longmapsto f(x) \\ \quad * \longmapsto * \end{array} \right)$$

(2) $F : \mathbb{C}\text{-Vect} \longrightarrow \mathbb{C}\text{-Vect}$

$$V \longmapsto V^{**}$$

$$(f : V \to W) \longmapsto (f^{**} : V^{**} \to W^{**})$$

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[dr, dashed, "h \circ f"'] & W \arrow[d, "h"] \\
& \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
W^* \arrow[r, "f^*"] \arrow[dr, dashed, "\varphi \circ f^*"'] & V^* \arrow[d, "\varphi"] \\
& \mathbb{C}
\end{tikzcd}$$

$f^* : W^* \to V^*$

$f^*(h) = h \circ f$

$$f^{**}(\varphi)(h) = (\varphi \circ f^*)(h) = \varphi(f^*(h))$$
$$= \varphi(h \circ f)$$