Category theory exercise sheet 4

October 2022

1 Category theory

1. Let **Cat** be the category of small categories and **Set** the category of sets. Consider the functor $\mathsf{Ob}\colon \mathbf{Cat} \to \mathbf{Set}$ which assigns to any small category $\mathcal{C}$ the set of its objects, and to a functor $F\colon \mathcal{C} \to \mathcal{D}$ the underlying function between the set of objects. We look for a left and right adjoint to Ob, if they exist.

   (a) We can assign to any set X a category $d(X)$, called the *descrete category* determined by X, whose set of objects is X and whose sets of morphisms are $\mathsf{Hom}(X, X) = \{\mathrm{id}_X\}$ and $\mathsf{Hom}(X, Y) = \emptyset$ if $X \neq Y$. Prove that $d(X)$ is indeed a category and that this assignment extends to a functor $d\colon \mathbf{Set} \to \mathbf{Cat}$.

   (b) We can assign to any set X a category $c(X)$, called the *convex category* determined by X, whose set of objects is X and whose sets of morphisms are always 1-point sets, i.e. $\mathsf{Hom}(X, Y) = \{*\}$ for all $X, Y \in \mathbf{Set}$. Prove that $c(X)$ is indeed a category and that this assignment extends to a functor $c\colon \mathbf{Set} \to \mathbf{Cat}$.

   Prove that $d \dashv \mathsf{Ob}$ and $\mathsf{Ob} \dashv c$.

2 Mathematics

2. Let $i\colon \mathbb{Z} \hookrightarrow \mathbb{R}$ be the inclusion of posets. Recall that we can regard a poset as a category. Prove that:

   (a) i is a functor of posets;

   (b) the ceiling and the floor function $\lceil - \rceil, \lfloor - \rfloor\colon \mathbb{R} \to \mathbb{Z}$ are functors of categories[1];

   (c) prove that $\lceil - \rceil$ is a left adjoint to i and that $\lfloor - \rfloor$ is a right adjoint to i.

[1] Recall that the function $\lceil - \rceil\colon \mathbb{R} \to \mathbb{Z}$ assign to $x \in \mathbb{R}$ the smallest integer greater or equal than x, while $\lfloor - \rfloor\colon \mathbb{R} \to \mathbb{Z}$ assigns to $x \in \mathbb{R}$ the greatest integer smaller or equal than x.

3. Consider $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$ the subset functor. For any set $X \in \mathbf{Set}$, we can define the maps

$$\eta_X\colon X \longrightarrow \mathcal{P}(X)$$

and

$$\mu_x\colon \mathcal{P}(\mathcal{P}(X)) \longrightarrow \mathcal{P}(X),$$

where $\eta_X(a) := \{a\}$ for any $a \in X$, and $\mu_X(B) = \cup_{A \in B} A$ for all $B \subseteq \mathcal{P}(X)$.

(a) Prove that $\eta = \{\eta_X\}_{X \in \mathbf{Set}}$ and $\mu = \{\mu_X\}_{X \in \mathbf{Sets}}$ define natural tranformations.

(b) Prove that $(\mathcal{P}, \mu, \eta)$ is a monad on $\mathbf{Set}$.

3 Logic

4. Let X be a set and $\Omega := \{T, F\}$ (T stands for True, F for False). We endow Ω of the structure of a poset by declaring that $F \leq T$. Then $\Omega^X := \mathsf{Hom}_{\mathbf{Sets}}(X, \Omega)$ is a partially ordered set, where

$$P \leq Q \iff P(x) \leq Q(x) \quad \forall x \in X.$$

Observe that if we regard an element $P \in \Omega^X$ as a proposition, then $P \leq Q$ means that P implies Q.

(a) There exist two functors $\forall_x, \exists_x\colon \Omega^X \to \Omega$, where

$$\forall_x(P) = T \text{ if and only if } P(x) = T \ \forall x \in X$$

and

$$\exists_x(P) = T \text{ if and only if } \exists x \in X \text{ such that } P(x) = T$$

.

(b) There exists a functor $\Delta\colon \Omega \to \Omega^X$ which sends an element of Ω to the constant function at that element.

Prove that $\exists_x \dashv \Delta$ and $\Delta \dashv \forall_x$.