Ultrafilters and Non-Standard Models of Arithmetic

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September 2022

1 Filters

Definition 1.0.1. A filter on a set X is a subset $\mathcal{F} \subseteq \mathcal{P}(X)$ which decides which subsets should be considered *large*, satisfying

1. $X \in \mathcal{F}$ (X is large)
2. $\emptyset \notin \mathcal{F}$ (the emptyset is not large)
3. If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$ (supersets of large sets are large)
4. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ (large sets have large intersection)

Example 1.0.2.

- The trivial filter: $\mathcal{F} = \{X\}$
- The cofinite filter (X infinite): $\mathcal{F} = \{A \subseteq X \mid X \setminus A \text{ is finite}\}$
- The principal filter generated by $x \in X$: $\mathcal{F} = \{A \subseteq X \mid x \in A\}$

Note that filters have the *finite intersection property* (FIP):

$$\text{If } A_1, \dots, A_n \in \mathcal{F}, \text{ then } \bigcap_i A_i \neq \emptyset$$

Lemma 1.0.3. *Any subset $S \subseteq \mathcal{P}(X)$ with the FIP has a minimal filter containing it, the filter generated by S*

Proof. Let S' be the closure of S under finite intersections, i.e. the smallest superset of S satisfying 4. Then $\mathcal{F} = \{A \subseteq X \mid \exists B \in S', B \subseteq A\}$ is a filter. S' cannot have introduced the $\emptyset$ by the FIP, and neither could adding supersets, so 2 remains true of $\mathcal{F}$. If $U, V \in \mathcal{F}$, then there are $U', V' \in S'$ with $U' \subseteq U, V' \subseteq V$, so $U \cap V$ is a superset of $U' \cap V'$, which is in S', so $U \cap V \in \mathcal{F}$. 1 and 3 are clear. □

Definition 1.0.4. A filter $\mathcal{F}$ on X is an *ultrafilter* if every subset is either large or *co-large*, i.e.

5. For any $A \subseteq X$, either $A \in F$ or $X \setminus A \in F$.

Principal filters are ultrafilters (since $x \notin A \Leftrightarrow x \in X \setminus A$) but the cofinite filter is not. For example, the subset $\{0, 2, 4, 6 \dots\}$ of even naturals is neither cofinite or finite, so it is neither large, nor co-large in the cofinite filter on $\mathbb{N}$. Are there any non-principal ultrafilters on an infinite set? Yes, classically. The difficulty in constructing them lies in deciding which of the infinite, but not cofinite, sets are large and which aren't (the cofinite sets are always present in an infinite ultrafilter).

Lemma 1.0.5. *$\mathcal{F}$ is an ultrafilter iff $\bigcup_{i=0}^{n} A_i \in \mathcal{F}$ implies that $A_i \in \mathcal{F}$ for some i (a large set cannot be a finite union of small sets).*

Proof. Suppose $\bigcup_{i=0}^{n} A_i \in \mathcal{F}$, but there is no i with $A_i \in \mathcal{F}$. With $B_i = X \setminus A_i$, every such B_i is in $\mathcal{F}$ by the ultrafilter property (5) and therefore $\bigcap_{i=0}^{n} B_i \in \mathcal{F}$ by repeated application of (4). This set is the complement of $\bigcup_{i=0}^{n} A_i$, so they cannot both be in the filter without contradicting (5). Hence there must be some $A_i \in \mathcal{F}$. □

Lemma 1.0.6. *(Ultrafilter lemma). Every filter is contained in an ultrafilter.*

Proof. We apply Zorn's lemma to the poset of filters on X containing $\mathcal{F}$. Observe that

- Given a chain of filters $\mathcal{F} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots$, their union is a filter. Axioms (1) - (3) hold for any union of filters, and 4 holds, since for any $A, B \in \bigcup_{i=0}^{\infty} \mathcal{F}_i$, $A \in \mathcal{F}_j, B \in \mathcal{F}_k$ for some j, k, so if $\mathcal{F}_j \subseteq \mathcal{F}_k$ then A and B both belong to the filter $\mathcal{F}_k$, and so does $A \cap B$.
- A maximal filter is an ultrafilter. If a filter $\overline{\mathcal{F}}$ is not an ultrafilter, then there is some A with $A \notin \overline{\mathcal{F}}$ and $X \setminus A \notin \overline{\mathcal{F}}$. But $\overline{\mathcal{F}} \cup \{A\}$ has the FIP. Any finite intersection involving A can be written $A \cap \bigcap_i U_i$ with each $U_i \in \overline{\mathcal{F}}$. But if this is empty, then $\bigcap_i U_i \subseteq X \setminus A$, contradicting that $\overline{\mathcal{F}}$ is a filter by 3. Hence $\overline{\mathcal{F}}$ is properly contained in the filter generated by $\overline{\mathcal{F}} \cup \{A\}$, so $\overline{\mathcal{F}}$ cannot be maximal.

Zorn's lemma therefore says that there is a maximal such filter among those containing $\mathcal{F}$, since any chain in the poset of such filters is upper bounded by the union filter of the chain. That maximal filter is an ultrafilter containing $\mathcal{F}$. □

Lemma 1.0.7. *Every non-principal ultrafilter contains the cofinite filter.*

Proof. A restatement of this lemma says that non-principal ultrafilters do not have finite sets - the ultrafilter condition forces it to have the complements. Suppose it had a finite set A, and consider the intersection $\bigcap_{a \in A} X \setminus \{a\}$. Every $X \setminus \{a\}$ is in $\mathcal{F}$, since their complements are singletons and $\mathcal{F}$ is non-principal, so the intersection is also $\mathcal{F}$, but that's just $X \setminus A$, contradicting $A \in \mathcal{F}$. □

References

[1] A. Kruckman, *Notes on Ultrafilters.* https://math.berkeley.edu/~kruckman/ultrafilters.pdf

[2] "Rising Entropy", *The Ultra Series.* https://risingentropy.com/the-ultra-series-guide/